

# Oscillating Rim Hook Tableaux and Colored Matchings

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## Abstract

Motivated by the question of finding a type  $B$  analogue of the bijection between oscillating tableaux and matchings, we find a correspondence between oscillating  $m$ -rim hook tableaux and  $m$ -colored matchings, where  $m$  is a positive integer. An oscillating  $m$ -rim hook tableau is defined as a sequence  $(\lambda^0, \lambda^1, \dots, \lambda^{2n})$  of Young diagrams starting with the empty shape and ending with the empty shape such that  $\lambda^i$  is obtained from  $\lambda^{i-1}$  by adding an  $m$ -rim hook or by deleting an  $m$ -rim hook. Our bijection relies on the generalized Schensted algorithm due to White. An oscillating 2-rim hook tableau is also called an oscillating domino tableau. When we restrict our attention to two column oscillating domino tableaux of length  $2n$ , we are led to a bijection between such tableaux and noncrossing 2-colored matchings on  $\{1, 2, \dots, 2n\}$ , which are counted by the product  $C_n C_{n+1}$  of two consecutive Catalan numbers. A 2-colored matching is noncrossing if there are no two arcs of the same color that are intersecting. We show that oscillating domino tableaux with at most two columns are in one-to-one correspondence with Dyck path packings. A Dyck path packing of length  $2n$  is a pair  $(D, E)$ , where  $D$  is a Dyck path of length  $2n$ , and  $E$  is a dispersed Dyck path of length  $2n$  that is weakly covered by  $D$ . So we deduce that Dyck path packings of length  $2n$  are counted by  $C_n C_{n+1}$ .

**Keywords:** Oscillating  $m$ -rim hook tableau,  $m$ -colored matching, lattice path, bijection, Dyck path packing

**AMS Classification:** 05A15, 05A18

## 1 Introduction

The objective of this paper is to provide a rim hook analogue of the correspondence between oscillating tableaux and matchings given by Chen, Deng, Du, Stanley and Yan [3]. We show that there is a one-to-one correspondence between oscillating  $m$ -rim hook tableaux and  $m$ -colored matchings. The construction of our bijection relies on the generalized Schensted algorithm for rim hook tableaux introduced by White [8].

We shall pay special attention to the case of oscillating domino (2-rim hook) tableaux with at most two columns. In this case, our main result reduces to a bijection between oscillating domino tableaux with at most two columns and noncrossing 2-colored matchings. Bear in mind that, a noncrossing 2-colored matching is not meant to be a noncrossing matching with two colors, but a matching that does not contain crossing edges that are of the same color. On the other hand, we find a correspondence between oscillating domino tableaux with at most two columns and Dyck path packings. A Dyck path packing can be viewed as a dispersed Dyck path

$E$  weakly covered by a Dyck path  $D$ , where a dispersed Dyck path is defined as non-overlapping Dyck paths connected by some horizontal steps on the  $x$ -axis. See Figure 1.1 for an illustration.

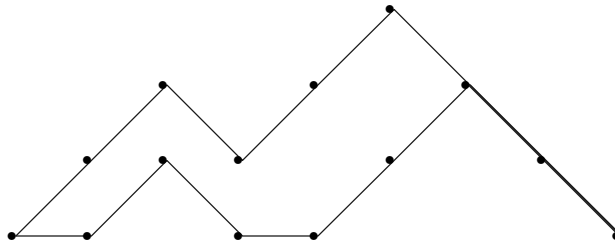


Figure 1.1: A Dyck path packing.

So we are led to a bijection between Dyck path packings and noncrossing 2-colored matchings. It is easy to check that noncrossing 2-colored matchings with  $2n$  vertices are counted by the product of two Catalan numbers, that is,  $C_n C_{n+1}$ , where

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Notice that there are several combinatorial objects that are enumerated by the number  $C_n C_{n+1}$ , such as walks within the first quadrant of  $\mathbb{Z}^3$  starting at  $(0,0,0)$  and consisting of  $n$  steps taken from  $\{(-1,0,0), (0,-1,1), (0,1,0), (1,0,-1)\}$ , see Bostan and Kauers [1], walks within the first quadrant of  $\mathbb{Z}^2$  starting at  $(0,0)$ , ending on the  $x$ -axis and consisting of  $2n$  steps taken from  $\{(-1,0), (-1,1), (1,-1), (1,0)\}$ , see Bousquet-Mélou and Mishna [2], alternating Baxter permutations of length  $2n+1$ , see Cori, Dulucq and Viennot [4], and walks within the first quadrant of  $\mathbb{Z}^2$  starting and ending at  $(0,0)$  and consisting of  $2n$  steps taken from  $\{(-1,0), (0,-1), (0,1), (1,0)\}$ , see Guy [5]. We also find a correspondence between noncrossing 2-colored matchings and Guy's walks. It would be interesting to establish further connections between packings of Dyck paths and other combinatorial structures.

Let us give an overview of some definitions. A partition of an integer  $n$  is a sequence  $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$  of nonincreasing positive integers such that  $\lambda_1 + \lambda_2 + \dots + \lambda_\ell = n$ . We can also represent a partition by its Young diagram, i.e., a left-justified array of cells (or, squares) with  $\lambda_i$  cells in row  $i$  for  $1 \leq i \leq \ell$ . For example, Figure 1.2 is the Young diagram of the partition  $(5, 4, 2, 2)$ .

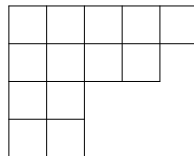


Figure 1.2: The Young diagram of  $(5, 4, 2, 2)$

To define the rim hooks of a partition, we note that the outside border of a partition  $\lambda$  is a collection of cells not in  $\lambda$  but immediately below or to the right of  $\lambda$ ; or in the first row and to

the right of  $\lambda$ ; or in the first column and below  $\lambda$ . For example, in Figure 1.3 the shaded area illustrates the outside border of  $\lambda = (5, 4, 2, 2)$ .

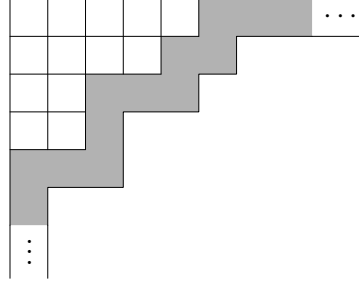


Figure 1.3: The outside border of  $(5, 4, 2, 2)$

Let  $\alpha$  be a set of contiguous cells in the outside border of  $\lambda$ . We say that  $\alpha$  is a rim hook outside  $\lambda$  if the shape  $\mu = \lambda \cup \alpha$  is the Young diagram of a partition. For example, in Figure 1.4, it can be seen that among the three sets of contiguous cells in the outside border of  $\lambda$ , there is only one rim hook outside  $\lambda$ , which is the third diagram. If  $\alpha$  has  $m$  cells, then we write  $|\alpha| = m$

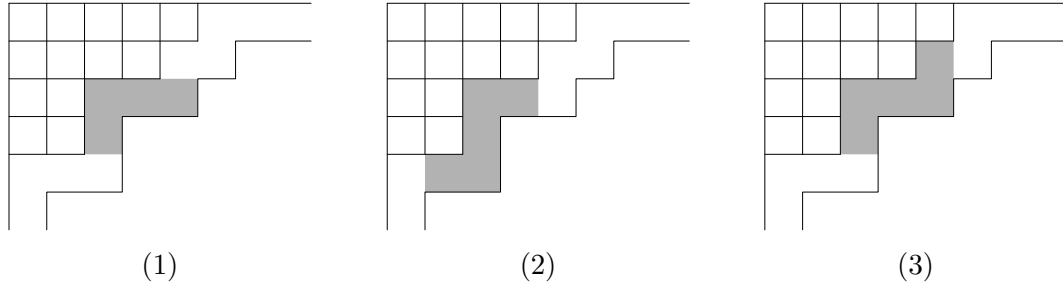


Figure 1.4: Contiguous cells in the outside border

and call  $\alpha$  an  $m$ -rim hook outside  $\lambda$ . If  $\alpha$  is a rim hook outside  $\lambda$  and  $\mu = \lambda \cup \alpha$ , then we call  $\alpha$  an outer rim hook of  $\mu$ , and we write  $\mu - \alpha$  to mean  $\lambda$ .

We can now introduce the notion of oscillating  $m$ -rim hook tableaux. An oscillating  $m$ -rim hook tableau of length  $2n$  can be defined as a sequence  $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$  of Young diagrams such that  $\lambda^0 = \lambda^{2n} = \emptyset$ , and for  $1 \leq i \leq 2n$ ,  $\lambda^i$  is obtained from  $\lambda^{i-1}$  either by adding an  $m$ -rim hook outside  $\lambda^{i-1}$  or by deleting an outer  $m$ -rim hook of  $\lambda^{i-1}$ . For example, Figure 1.5 is an illustration of an oscillating 3-rim hook tableau.

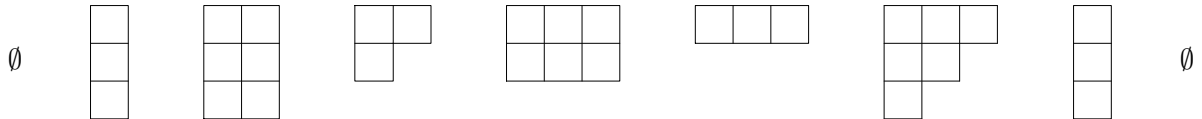


Figure 1.5: An oscillating 3-rim hook tableau

When  $m = 1$ , an oscillating  $m$ -rim hook tableau is an ordinary oscillating tableau. An

oscillating 2-rim hook tableau will be also called an oscillating domino tableau. We shall use  $r(\lambda)$  (resp.,  $c(\lambda)$ ) to denote the maximum number of rows (resp., columns) of the shape  $\lambda^i$  appearing in  $\lambda$  for  $0 \leq i \leq 2n$ .

The main objective of this paper is to show that there is a one-to-one correspondence between oscillating  $m$ -rim hook tableaux and  $m$ -colored matchings. An  $m$ -colored matching  $M$  on  $[2n]$  is a matching on  $[2n]$  with each arc assigned one of  $m$  colors, say,  $c_1, c_2, \dots, c_m$ . For example, Figure 1.6 gives a 2-colored matching, where we use solid lines to represent arcs assigned the color  $c_1$ , and dotted lines to represent arcs assigned the color  $c_2$ . A  $k$ -crossing of  $M$  is a  $k$ -subset

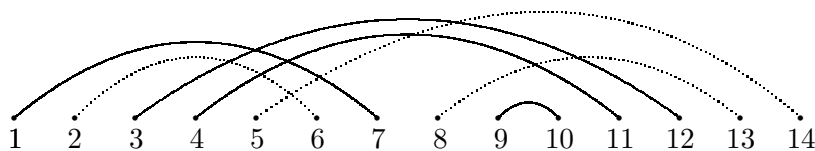


Figure 1.6: A 2-colored matching

$\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$  of arcs of the same color such that  $i_1 < i_2 < \dots < i_k < j_1 < j_2 < \dots < j_k$ . Similarly, we define a  $k$ -nesting of  $M$  as a  $k$ -subset  $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$  of arcs of the same color such that  $i_1 < i_2 < \dots < i_k < j_k < \dots < j_2 < j_1$ . Denote by  $\text{cr}(M)$  (resp.,  $\text{ne}(M)$ ) the maximal number  $k$  such that  $M$  has a  $k$ -crossing (resp.,  $k$ -nesting). We say that  $M$  is  $k$ -noncrossing (resp.,  $k$ -nonnesting) if  $M$  has no  $k$ -crossing (resp.,  $k$ -nesting). A 2-noncrossing (or, 2-nonnesting)  $m$ -colored matching is called a noncrossing (or, nonnesting)  $m$ -colored matching.

Our bijection can be used to characterize the crossing number and the nesting number of an  $m$ -colored matching in terms of the maximum number of columns and the maximum number of rows of shapes in the corresponding oscillating  $m$ -rim hook tableau. The construction of our bijection is based on the generalized Schensted algorithm for rim hook tableaux due to White [8].

This paper is organized as follows. We shall give a brief review of White's algorithm in Section 2. Based on this algorithm, we give a description of the bijection between oscillating  $m$ -rim hook tableaux and  $m$ -colored matchings in Section 3. Section 4 is concerned with oscillating domino tableaux with at most two columns and noncrossing 2-colored matchings. We show that such tableaux are in one-to-one correspondence with Dyck path packings. We also give a bijection between noncrossing 2-colored matchings and Guy's walks.

## 2 The generalized Schensted algorithm

To give a combinatorial proof of the orthogonality of the characters of the symmetric group  $S_n$ , White [8] extended the ordinary Schensted algorithm [6] to rim hook tableaux. For our purpose, we shall be concerned with only a special case of White's construction when all rim hooks and hooks are restricted to  $m$ -rim hooks and  $m$ -hooks. This version of White's algorithm has been further studied by Stanton and White [7]. As remarked by White [8], when  $m = 2$ , White's

algorithm reduces to a one-to-one correspondence between elements of the hyperoctahedral group and pairs of domino tableaux of the same shape, which was first obtained by Lusztig. We shall adopt the notation and terminology in [7].

For two partitions  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\mu = (\mu_1, \mu_2, \dots)$ , we write  $\mu \subseteq \lambda$  if  $\mu_i \leq \lambda_i$  for all  $i$ . If  $\mu \subseteq \lambda$ , then the skew diagram of shape  $\lambda/\mu$  is defined as the set of cells obtained from  $\lambda$  by deleting the cells in  $\mu$ . For example, the shaded area in Figure 2.1 represents a skew diagram.

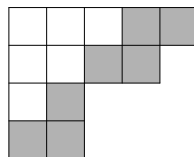


Figure 2.1: The skew diagram of shape  $(5, 4, 2, 2)/(3, 2, 1)$

Let  $i_1, i_2, \dots, i_n$  be  $n$  positive integers with  $i_1 < i_2 < \dots < i_n$ . An  $m$ -rim hook tableau  $P$  of shape  $\lambda$  on  $\{i_1, i_2, \dots, i_n\}$  is an assignment of  $i_1, i_2, \dots, i_n$  to the squares of  $\lambda$  with each integer appearing exactly  $m$  times such that the set  $\alpha$  of squares occupied by  $i_n$  is an outer  $m$ -rim hook of  $\lambda$ , and the tableau obtained from  $P$  by deleting the squares occupied by  $i_n$  is an  $m$ -rim hook tableau on  $\{i_1, i_2, \dots, i_{n-1}\}$ . We say that  $i_1, i_2, \dots, i_n$  are the contents of  $P$ , and write  $\text{content}(P) = \{i_1, i_2, \dots, i_n\}$ . Similarly, we can define skew  $m$ -rim hook tableaux of shape  $\lambda/\mu$ . Figure 2.2 is an illustration of a 4-rim hook tableau.

1	1	1	1	4	4
2	3	3	4	4	
2	3	5			
2	3	5			
2	5	5			

Figure 2.2: A 4-rim hook tableau

Recall that an  $m$ -hook is the Young diagram corresponding to a partition  $(t, 1, 1, \dots, 1)$  of  $m$ , where  $1 \leq t \leq m$ . An  $m$ -hook tableau is an  $m$ -rim hook tableau whose shape is an  $m$ -hook. Figure 2.3 gives a 4-hook tableau of shape  $(2, 1, 1)$ .

3	3
3	
3	

Figure 2.3: A 4-hook tableau

Let  $P$  be an  $m$ -rim hook tableau, and  $H$  be an  $m$ -hook tableau. The generalized Schensted algorithm is an algorithm to generate an  $m$ -rim hook tableau by inserting  $H$  to  $P$ . When  $m = 1$ , it reduces to the usual Schensted algorithm. To describe the generalized Schensted algorithm, we need recall more definitions.

Let  $\alpha$  be a set of contiguous cells contained in the outside border of  $\lambda$ . The head (resp., tail) of  $\alpha$  is the upper rightmost (resp., lower leftmost) square in  $\alpha$ . The head (resp., tail) of  $\alpha$  is said to be illegal with respect to  $\lambda$  if the cell above (resp., to the left) the head (resp., tail) is in the outside border of  $\lambda$ . For example, in Figure 1.4 the contiguous cells in (1) have an illegal head, whereas the contiguous cells in (2) have an illegal tail. Clearly, if  $\alpha$  has neither an illegal head nor an illegal tail with respect to  $\lambda$ , then  $\alpha$  is a rim hook outside  $\lambda$ . Even though  $\alpha$  is not a rim hook outside  $\lambda$ , it may be a rim hook outside another partition. We shall say that  $\alpha$  is a rim hook if it is a rim hook outside a certain partition.

Let  $\sigma$  be a rim hook contained in the outside border of  $\lambda$ . Let  $slitherup(\lambda, \sigma)$  denote the rim hook contained in the outside border of  $\lambda$ , whose tail is adjacent to the head of  $\sigma$ . Similarly, we can define  $slitherdown(\lambda, \sigma)$ . Figure 2.4 gives illustrations of these two operations.

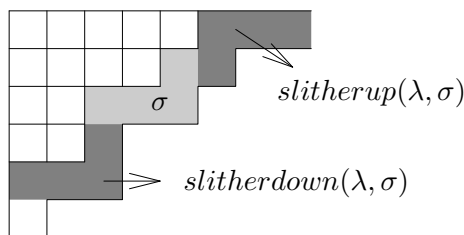


Figure 2.4:  $slitherup(\lambda, \sigma)$  and  $slitherdown(\lambda, \sigma)$

If  $\alpha$  is a collection of cells which are not necessarily contiguous, then  $bumpout(\alpha)$  is the collection of cells directly below and to the right of a single cell in  $\alpha$ . If  $\sigma$  and  $\tau$  are two distinct rim hooks outside  $\lambda$  such that  $\sigma \cap \tau \neq \emptyset$ , then define  $\sigma[\tau]$  as a rim hook outside  $\lambda \cup \tau$ , that is,

$$\sigma[\tau] = (\sigma - \sigma \cap \tau) \cup bumpout(\sigma \cap \tau).$$

See Figure 2.5 for an illustration.

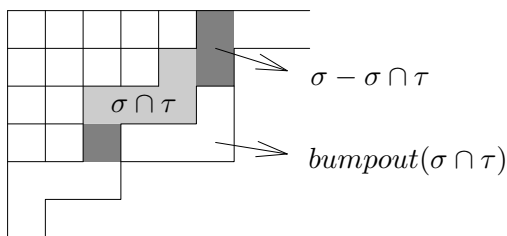


Figure 2.5:  $\sigma[\tau]$

Let  $U$  be an  $m$ -rim hook tableau of shape  $\lambda$ , and  $V$  a skew  $m$ -rim hook tableau of shape  $\mu/\omega$ . The pair  $(U, V)$  is called an overlapping pair if  $\sigma = \lambda/\omega$  is a rim hook outside  $\omega$ , and any content in  $U$  is smaller than any content in  $V$ .

Given an overlapping pair  $(U, V)$ , Stanton and White [7] defined an operator  $A$  acting on the pair  $(U, V)$ , which generates an overlapping pair  $(U^1, V^1) = A(U, V)$ . To be precise, the operator  $A$  can be described as follows. Suppose that  $r$  is the minimum content in  $V$ , and  $\tau$

is the  $m$ -rim hook containing  $r$ . For a rim hook  $\alpha$  and an integer  $s$ , let  $\alpha(s)$  denote the rim hook  $\alpha$  with each cell of  $\alpha$  filled with  $s$ . The skew  $m$ -rim hook tableau  $V^1$  is obtained from  $V$  by removing  $\tau(r)$ . Then  $U^1$  can be constructed depending on how the shapes  $\sigma$  and  $\tau$  overlap. Note that  $\sigma = \lambda/\omega$ , where  $\lambda$  is the shape of  $U$  and  $\mu/\omega$  is the shape of  $V$ . There are three cases.

Case 1:  $\sigma \cap \tau = \emptyset$ . Set  $U^1 = U \cup \tau(r)$ .

Case 2:  $\sigma \cap \tau \neq \emptyset$  and  $\sigma \neq \tau$ . Set  $U^1 = U \cup \tau[\sigma](r)$ .

Case 3:  $\sigma = \tau$ . In this case, we construct a sequence  $(\tau = \tau_0, \tau_1, \tau_2, \dots)$  of  $m$ -rim hooks contained in the outside border of  $\lambda$ , where  $\tau_i = \text{slitherup}(\lambda, \tau_{i-1})$  for  $i \geq 1$ . Assume that  $m_0$  is the smallest integer such that the head of  $\tau_{m_0}$  is legal with respect to  $\lambda$ . Set  $U^1 = U \cup \tau_{m_0}(r)$ .

It is not difficult to check that  $A(U, V)$  is an overlapping pair, see [8]. Let

$$A^n(U, V) = A^{n-1}(A(U, V)),$$

and write

$$(U^n, V^n) = A^n(U, V).$$

Assume that  $n_0$  is the smallest integer such that  $V^{n_0}$  is empty. Then, define

$$\text{Combine}(U, V) = U^{n_0}.$$

We can now state the rim hook insertion algorithm. Suppose that  $P$  is an  $m$ -rim hook tableau of shape  $\lambda$  with contents not containing  $r$ , and  $H$  is an  $m$ -hook tableau with content  $r$ . The rim hook insertion algorithm gives an  $m$ -rim hook tableau by inserting  $H$  to  $P$ , denoted  $P \leftarrow H$ . To obtain  $P \leftarrow H$ , we need to define an overlapping pair  $(U, V)$ . Let  $P = P_1 \cup P_2$ , where  $P_1$  (resp.,  $P_2$ ) is the  $m$ -rim hook tableau contained in  $P$  with contents smaller than (resp., greater than)  $r$ . We set  $V = P_2$ .

To define  $U$ , we assume that  $\tau$  is the shape of  $H$ , and  $\lambda'$  is the shape of  $P_1$ . Let  $(\tau_0, \tau_1, \tau_2, \dots)$  be a sequence of  $m$ -rim hooks, where  $\tau_0 = \tau$  and  $\tau_i = \text{slitherup}(\emptyset, \tau_{i-1})$  for  $i \geq 1$ . Moreover, we assume that  $m_0$  is the smallest integer such that  $\tau_{m_0}$  has no intersection with  $\lambda'$ . We still need another sequence  $(\sigma_0, \sigma_1, \sigma_2, \dots)$  of  $m$ -rim hooks contained in the outside border of  $\lambda'$ , where  $\sigma_0 = \tau_{m_0}$  and  $\sigma_i = \text{slitherup}(\lambda', \sigma_{i-1})$  for  $i \geq 1$ . Assume that  $n_0$  is the smallest integer such that  $\sigma_{n_0}$  has a legal head with respect to  $\lambda'$ . Then  $U$  is set to be  $P_1 \cup \sigma_{n_0}(r)$ .

Based on the  $m$ -rim hook tableaux  $U$  and  $V$ , the  $m$ -rim hook tableau  $P \leftarrow H$  can be defined as  $\text{Combine}(U, V)$ . It can be shown that the above rim hook insertion algorithm is invertible, see [8].

The rim hook insertion algorithm leads to a one-to-one correspondence between  $m$ -hook permutations and pairs of  $m$ -rim hook tableaux of the same shape. Let  $i_1, i_2, \dots, i_n$  be  $n$  positive integers with  $i_1 < i_2 < \dots < i_n$ . An  $m$ -hook permutation on  $\{i_1, i_2, \dots, i_n\}$  is a permutation of  $n$   $m$ -hook tableaux such that the contents of these  $m$ -hook tableaux read off from left to right form a permutation on  $\{i_1, i_2, \dots, i_n\}$ . For example, Figure 2.6 illustrates a 4-hook permutation on  $\{1, 4, 6, 9\}$ .

As remarked in [7], an  $m$ -hook permutation on  $[n]$  can be viewed as an element of the wreath product  $C_m \wr S_n$ , where  $C_m$  is the cyclic group of order  $m$ , or equivalently, as an  $m$ -colored permutation on  $[n]$  in the sense that each element in the permutation is assigned one of the colors  $c_1, c_2, \dots, c_m$ .





and

$$\lceil c(\lambda)/m \rceil = \text{cr}(\phi(\lambda)). \quad (3.2)$$

*Proof.* We first describe the bijection  $\phi$  from oscillating  $m$ -rim hook tableaux of length  $2n$  to  $m$ -colored matchings on  $[2n]$ . Let

$$\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$$

be an oscillating  $m$ -rim hook tableau. We shall recursively define a sequence

$$(M_0, T_0), (M_1, T_1), \dots, (M_{2n}, T_{2n}),$$

where  $M_i$  is a set of  $m$ -colored arcs, and  $T_i$  is an  $m$ -rim hook tableau of shape  $\lambda^i$ . Let  $M_0$  be the empty set, and let  $T_0$  be the empty tableau. To obtain  $(M_i, T_i)$  for  $i \geq 1$ , we have the following two cases.

Case 1:  $\lambda^i$  is obtained from  $\lambda^{i-1}$  by adding an  $m$ -rim hook outside  $\lambda_{i-1}$ . In this case, let  $M_i = M_{i-1}$  and let  $T_i$  be the  $m$ -rim hook tableau obtained from  $T_{i-1}$  by filling the  $m$ -rim hook  $\lambda^i/\lambda^{i-1}$  with the element  $i$ .

Case 2:  $\lambda^i$  is obtained from  $\lambda^{i-1}$  by deleting an outer  $m$ -rim hook of  $\lambda_{i-1}$ . In this case, let  $T_i$  be the  $m$ -rim hook tableau of shape  $\lambda^i$  such that  $T_{i-1}$  is obtained from  $T_i$  by inserting an  $m$ -hook tableau  $H$ . Note that  $H$  is uniquely determined since the generalized Schensted algorithm is invertible. Suppose that  $H$  has shape  $(t, 1, 1, \dots, 1)$  and  $\text{content}(H) = \{j\}$ . It is easy to see that  $j < i$ . Let  $M_i$  be the set obtained from  $M_{i-1}$  by adding an arc  $(j, i)$  colored with  $c_t$ .

From the above construction, it is easy to check that  $M_{2n}$  is an  $m$ -colored matching on  $[2n]$ . We set  $\phi(\lambda) = M_{2n}$ .

The reverse map  $\psi$ , from  $m$ -colored matchings to oscillating  $m$ -rim hook tableaux, can be described as follows. Let  $M$  be an  $m$ -colored matching on  $[2n]$ . We shall construct a sequence  $T(M) = (T_0, T_1, \dots, T_{2n})$  of  $m$ -rim hook tableaux by a recursive procedure.

Let  $T_{2n}$  be the empty tableau. For  $j \leq 2n - 1$ ,  $T_j$  can be constructed based on the following two cases.

Case 1:  $j + 1$  is the right-hand endpoint of an arc  $(i, j + 1)$  colored with  $c_t$ . In this case, let  $T_j$  be the  $m$ -rim hook tableau obtained from  $T_{j+1}$  by inserting an  $m$ -hook tableau of shape  $(t, 1, \dots, 1)$  filled with  $i$ .

Case 2:  $j + 1$  is the left-hand endpoint of an arc  $(j + 1, k)$ . Note that in this case  $j + 1$  is the largest content of  $T_{j+1}$ . Let  $T_j$  be the  $m$ -rim hook tableau obtained from  $T_{j+1}$  by deleting the  $m$ -rim hook filled with  $j + 1$ .

Set  $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$ , where  $\lambda^i$  is the shape of  $T_i$ . From the above procedure, it can be seen that  $\lambda$  is an oscillating  $m$ -rim hook tableau. Define  $\psi(M) = \lambda$ . It is not hard to check that  $\psi$  is the reverse map of  $\phi$ .

It remains to prove the relations (3.1) and (3.2). For a  $m$ -colored matching  $M$ , let  $T(M) = (T_0, T_1, \dots, T_{2n})$  be the sequence of  $m$ -rim hook tableaux constructed from  $M$ . We now give a recursive procedure to generate a sequence of  $m$ -hook permutations  $(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{2n})$  from the sequence  $T(M)$  of  $m$ -rim hook tableaux. Let  $\mathcal{H}_{2n}$  be the empty hook permutation. Suppose that

the  $m$ -hook permutation  $\mathcal{H}_i$  has been constructed. To generate  $\mathcal{H}_{i-1}$ , we consider the following two cases.

Case 1:  $T_{i-1}$  is obtained from  $T_i$  by inserting an  $m$ -Hook tableau  $H_j$ . In this case, let  $\mathcal{H}_{i-1} = \mathcal{H}_i H_j$ .

Case 2:  $T_i$  is obtained from  $T_{i-1}$  by adding an  $m$ -rim hook filled with  $i$ . In this case, let  $\mathcal{H}_{i-1}$  be the hook permutation obtained from  $\mathcal{H}_i$  by deleting the  $m$ -hook tableau with content  $i$ .

We claim that  $T_i$  is the insertion tableau of  $\mathcal{H}_i$ . This can be shown by induction. Clearly, the claim holds for  $i = 2n$ . Suppose that it is true for  $i$ , where  $1 < i \leq 2n$ . We shall show that the claim holds for  $i - 1$ . If  $\mathcal{H}_{i-1} = \mathcal{H}_i H_j$ , then the statement is obvious. So it suffices to consider the case when  $\mathcal{H}_{i-1}$  is the hook permutation obtained from  $\mathcal{H}_i$  by deleting the  $m$ -hook tableau with content  $i$ . Observe that in this case,  $i$  must be the largest content appearing in the  $m$ -hook tableaux of  $\mathcal{H}_i$ . In view of Proposition 2.3, we see that  $T_{i-1}$  is the insertion tableau of  $\mathcal{H}_{i-1}$ .

To finish the proof of (3.1), we proceed to show that  $M$  has a  $k$ -crossing if and only if there exists an  $m$ -hook permutation  $\mathcal{H}_i$  ( $1 \leq i \leq 2n$ ) that contains a decreasing subsequence of length  $k$ . Suppose that  $\mathcal{H}_i = H_{i_1} H_{i_2} \cdots H_{i_{t_i}}$  ( $1 \leq i \leq 2n$ ) contains a decreasing subsequence of length  $k$ . By the construction of  $\mathcal{H}_i$ , there are  $t_i$   $m$ -colored arcs of  $M$  whose left-hand endpoints are  $\text{content}(H_{i_1}), \text{content}(H_{i_2}), \dots, \text{content}(H_{i_{t_i}})$ . For  $1 \leq s \leq t_i$ , let  $w_{is}$  be the right-hand endpoint of the arc with left-hand endpoint  $\text{content}(H_{i_s})$ . Again, by the construction of the sequence  $(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{2n})$ , it can be easily checked that

$$w_{i1} > w_{i2} > \cdots > w_{it_i}. \quad (3.3)$$

Denote by  $H_{is_1} H_{is_2} \cdots H_{is_k}$  the decreasing subsequence of  $\mathcal{H}_i$ . Clearly, the arcs

$$(\text{content}(H_{is_1}), w_{is_1}), \dots, (\text{content}(H_{is_k}), w_{is_k})$$

have the same color. Moreover, from (3.3) it follows that they form a  $k$ -crossing of  $M$ .

Finally, we need to consider the other direction of the above statement. Suppose that  $M$  has a  $k$ -crossing consisting the arcs

$$(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$$

with  $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$ . By the construction of  $\phi$ , it can be easily checked that  $i_1, \dots, i_k$  are contained in  $\text{content}(T_{j_1-1})$ . Consider the  $m$ -hook permutation  $\mathcal{H}_{j_1-1}$ . Replacing  $i$  with  $j_1 - 1$  in (3.3), we obtain that the subsequence of  $\mathcal{H}_{j_1-1}$  consisting of the  $m$ -hook tableaux with contents  $i_1, i_2, \dots, i_k$  is a decreasing subsequence of length  $k$ . This proves the above claim. Thus, relation (3.1) can be deduced from Theorem 2.2.

To prove (3.2), it suffices to show that  $M$  has a  $k$ -nesting if and only if there exists an  $m$ -hook permutation  $\mathcal{H}_i$  ( $1 \leq i \leq 2n$ ) that contains an increasing subsequence of length  $k$ . The proof is omitted since it is analogous to the case for  $k$ -crossings.  $\blacksquare$

Figure 3.1 gives an illustration of the bijection  $\phi$  for an oscillating domino tableau. In this case, the corresponding 2-colored matching is given in Figure 3.2, where we use solid lines to represent arcs assigned the color  $c_1$ , and dotted lines to represent arcs with color  $c_2$ . It can be

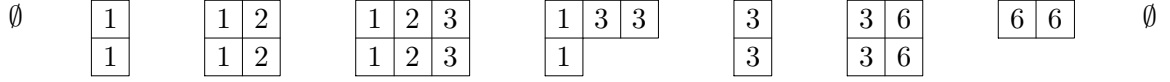


Figure 3.1: An illustration of the bijection  $\phi$

easily checked that

$$\lceil r(\lambda)/2 \rceil = \text{ne}(\phi(\lambda)) = 1 \quad \text{and} \quad \lceil c(\lambda)/2 \rceil = \text{cr}(\phi(\lambda)) = 2.$$

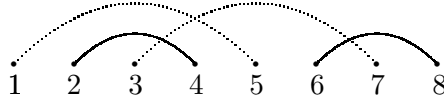


Figure 3.2: An example

The bijection in Theorem 3.1 implies a symmetry property for the crossing number and the nesting number of  $m$ -colored matchings. Let  $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$  be an oscillating  $m$ -rim hook tableau. The conjugate oscillating  $m$ -rim hook tableau of  $\lambda$ , denoted by  $\lambda'$ , is defined by  $(\mu^0, \mu^1, \dots, \mu^{2n})$ , where  $\mu^i$  is the conjugate of  $\lambda^i$ . By Theorem 3.1, we get an involution on  $m$ -colored matchings satisfying the following symmetry property.

**Corollary 3.2** *The crossing number  $\text{cr}(M)$  and the nesting number  $\text{ne}(M)$  have a symmetric distribution over all  $m$ -colored matchings on  $[2n]$ , that is, the number of  $m$ -colored matchings on  $[2n]$  with crossing number  $i$  and nesting number  $j$  equals the number of  $m$ -colored matchings on  $[2n]$  with crossing number  $j$  and nesting number  $i$ , for  $0 \leq i, j \leq n$ .*

We note that the above symmetry property can also be deduced from the bijection between oscillating tableaux and ordinary matchings, see Chen, Deng, Du, Stanley and Yan [3]. More precisely, an  $m$ -colored matching  $M$  can be considered as a sequence  $(M_1, M_2, \dots, M_m)$  of disjoint matchings on  $[2n]$ , where  $M_i$  is the set of arcs of  $M$  colored with  $c_i$ . It is obvious that  $M$  is  $k$ -noncrossing if and only if each  $M_i$  is  $k$ -noncrossing. Nevertheless, Theorem 3.1 does not seem to be a direct consequence of the correspondence for the case  $m = 1$ .

## 4 Oscillating domino tableaux

In this section, we introduce the structure of packings of Dyck paths and establish a connection with oscillating domino tableaux with each shape having at most two columns. By Theorem 3.1, we see that such oscillating domino tableaux are in one-to-one correspondence with noncrossing 2-colored matchings. We show that oscillating domino tableaux with at most two columns are in one-to-one correspondence with Dyck path packings. This means that there is a correspondence between Dyck path packings and noncrossing 2-colored matchings. It is easy to prove that the

number of noncrossing 2-colored matchings on  $[2n]$  equals  $C_n C_{n+1}$ , where  $C_n$  is the  $n$ -th Catalan number.

As will be seen, noncrossing 2-colored matchings are quite close to Guy's walks [5] in the sense that there is a simple bijection between these two structures. However, it should be mentioned that the problem of counting Dyck path packings does not seem to be as easy as counting noncrossing 2-colored matchings. Using the identity

$$\sum_{i=0}^n \binom{2n}{2i} C_i C_{n-j} = C_n C_{n+1}, \quad (4.1)$$

see [4], we obtain the following formula.

**Theorem 4.1** *For  $n \geq 0$ , the number of noncrossing 2-colored matchings on  $[2n]$  equals  $C_n C_{n+1}$ .*

Let  $O_{2n,2}$  be the set of oscillating domino tableaux  $\lambda$  of length  $2n$  with each shape appearing in  $\lambda$  having at most two columns. By Theorem 4.1, we see that the set  $O_{2n,2}$  is also counted by  $C_n C_{n+1}$ . We shall give a bijection between  $O_{2n,2}$  and Dyck path packings. A Dyck path of length  $2n$  is a lattice path in the plane from  $(0,0)$  to  $(2n,0)$  with steps  $(1,1)$  and  $(1,-1)$  that never passes below the  $x$ -axis. A dispersed Dyck path is a concatenation of Dyck paths and some horizontal steps  $(1,0)$  on the  $x$ -axis. In other words, a dispersed Dyck path of length  $2n$  is a lattice path from  $(0,0)$  to  $(2n,0)$  with up and down steps above the  $x$ -axis and with horizontal steps on the  $x$ -axis. We say that a dispersed Dyck path  $E$  is weakly covered by a Dyck path  $D$  if  $E$  never goes above  $D$ . If  $D$  is a Dyck path of length  $2n$  and  $E$  is a dispersed Dyck path of length  $2n$  that is weakly covered by  $D$ , then we say that the pair  $(D, E)$  is a packing of Dyck paths of length  $2n$ . Denote by  $P_{2n}$  the set of Dyck path packings of length  $2n$ .

**Theorem 4.2** *There is a bijection between the set  $O_{2n,2}$  of oscillating domino tableaux and the set  $P_{2n}$  of Dyck path packings.*

*Proof.* Let  $\mu = (\mu^0, \mu^1, \dots, \mu^{2n})$  be an oscillating domino tableau in  $O_{2n,2}$ , and let  $\lambda = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$  be the conjugate oscillating domino tableau of  $\mu$ , that is,  $\lambda^i$  is the conjugate of  $\mu^i$  ( $0 \leq i \leq 2n$ ). Since each shape  $\lambda^i$  has at most two rows, we may write  $\lambda^i = (u_i, v_i)$ , where  $u_i \geq v_i \geq 0$ . Clearly, both  $u_i + v_i$  and  $u_i - v_i$  are even. For  $0 \leq i \leq 2n$ , let

$$a_i = \frac{u_i + v_i}{2} \quad \text{and} \quad b_i = \frac{u_i - v_i}{2}.$$

Define two lattice paths  $D$  and  $E$  by setting

$$D = ((0, a_0), (1, a_1), \dots, (2n, a_{2n})) \quad \text{and} \quad E = ((0, b_0), (1, b_1), \dots, (2n, b_{2n})), \quad (4.2)$$

where a lattice path is represented by the lattice points.

We proceed to show that the map  $\alpha: \mu \longrightarrow (D, E)$  defined by the above construction is a bijection from  $O_{2n,2}$  to  $P_{2n}$ . As the first step, we prove that  $(D, E)$  is a packing of Dyck paths of length  $2n$ . It can be easily checked that  $a_0 = a_{2n} = 0$  and  $(i+1, a_{i+1}) - (i, a_i) = (1, 1)$  or  $(1, -1)$ .

Hence  $D$  is a Dyck path of length  $2n$ . Let us consider the possible values of  $(i+1, b_{i+1}) - (i, b_i)$ . There are two cases.

Case 1:  $\lambda^{i+1}$  is obtained from  $\lambda^i$  by adding or deleting a horizontal domino. In this case, it is easy to check that  $(i+1, b_{i+1}) - (i, b_i) = (1, 1)$  or  $(1, -1)$ .

Case 2:  $\lambda^{i+1}$  is obtained from  $\lambda^i$  by adding or deleting a vertical domino. In this case, both  $\lambda^i$  and  $\lambda^{i+1}$  have two rows with the same number of cells, that is,  $u_i = v_i$  and  $u_{i+1} = v_{i+1}$ . This implies that  $b_i = b_{i+1} = 0$  and  $(i+1, b_{i+1}) - (i, b_i) = (1, 0)$ . So we deduce that if there is a horizontal step on the path  $E$ , then it lies on the  $x$ -axis. It is clear that  $b_i \geq 0$ . Thus  $E$  is a dispersed Dyck path of length  $2n$ .

Since  $a_i \geq b_i$ , we see that  $E$  is weakly covered by  $D$ . Hence we conclude that  $(D, E)$  is a packing of Dyck paths of length  $2n$ .

The inverse map of  $\alpha$  can be described as follows. Let  $(D, E)$  be a Dyck path packing in  $P_{2n}$ . Write  $D$  and  $E$  in the forms as in (4.2). Set  $\lambda^i = (a_i + b_i, a_i - b_i)$ , and let  $\mu^i$  be the conjugate of  $\lambda^i$ . It is easy to check that  $(\mu^0, \mu^1, \dots, \mu^{2n})$  is an oscillating domino tableau belonging to  $O_{2n,2}$ . This completes the proof.  $\blacksquare$

For example, let  $\lambda$  be the conjugate of the oscillating domino tableau in Figure 3.1. The corresponding Dyck path packing is given in Figure 4.1.

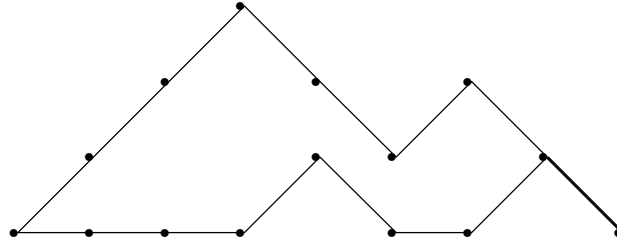


Figure 4.1: An example for Theorem 4.2

By Theorem 4.1 and Theorem 4.2, we obtain the following relations.

**Theorem 4.3** For  $n \geq 0$ ,

$$|O_{2n,2}| = |P_{2n}| = C_n C_{n+1}. \quad (4.3)$$

To conclude this paper, we give a bijection between noncrossing 2-colored matchings and Guy's walks. More precisely, a Guy's walk is defined to be a lattice walk within the first quadrant starting and ending at  $(0, 0)$  and consisting of the following steps

$$(-1, 0), (0, -1), (0, 1), (1, 0).$$

Figure 4.2 gives a Guy's walk with steps

$$\rightarrow \uparrow \uparrow \downarrow \rightarrow \uparrow \uparrow \rightarrow \downarrow \leftarrow \downarrow \leftarrow \leftarrow \uparrow \downarrow \downarrow.$$

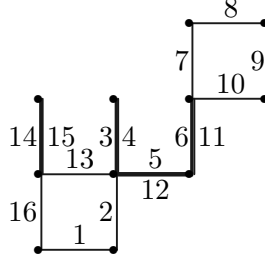


Figure 4.2: A Guy's walk

**Theorem 4.4** *There is a bijection between noncrossing 2-colored matchings on  $[2n]$  and Guy's walks with  $2n$  steps.*

*Proof.* We construct a bijection  $\beta$  from noncrossing 2-colored matchings on  $[2n]$  to Guy's walks with  $2n$  steps. Let  $M$  be a noncrossing 2-colored matching on  $[2n]$ . For any  $i$  in  $[2n]$ , we define a step  $s_i$  as follows:

$$s_i = \begin{cases} (1, 0), & \text{if } i \text{ is the left-hand endpoint of an arc with color } c_1; \\ (-1, 0), & \text{if } i \text{ is the right-hand endpoint of an arc with color } c_1; \\ (0, 1), & \text{if } i \text{ is the left-hand endpoint of an arc with color } c_2; \\ (0, -1), & \text{if } i \text{ is the right-hand endpoint of an arc with color } c_2. \end{cases}$$

Then define  $\beta(M)$  to be the walk  $s_1 s_2 \cdots s_{2n}$  starting at the origin.

We claim that  $\beta(M)$  is a walk in the first quadrant, starting and ending at the origin. For any  $i \in [2n]$ , consider the first  $i$  steps  $s_1, s_2, \dots, s_i$ . Since  $M$  is a noncrossing 2-colored matching, it is not hard to check that there are at least as many  $(1, 0)$  steps as  $(-1, 0)$  steps in  $\{s_1, s_2, \dots, s_i\}$ . Similarly, there are at least as many  $(0, 1)$  steps as  $(0, -1)$  steps in  $\{s_1, s_2, \dots, s_i\}$ . Hence  $\beta(M)$  is in the first quadrant. It is clear that  $\beta(M)$  terminates at the origin. So the claim holds. It is not difficult to see that the map  $\beta$  is invertible. This completes the proof. ■

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